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Statistics of a Whiteness Measure

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PREFACE

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A whiteness measure of a random number generator is defined as the sum of squares of all the off-zero elements of the sample covariance function of a finite segment of data of length K. The mean and variance of this whiteness measure are evaluated exactly, while its cumulative and exceedance distribution functions are determined by simulations. It is found that the variance of the whiteness measure must be broken into the two separate cases where K is even versus K is odd. This necessitates the analytic evaluation of some high-order moments in order to determine the variance exactly.

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

K	number of data points, (1)
x_k	k-th data value, (1)
IID	independent identically distributed
$E(\ )$	expectation of random variable, (1)
F	fourth moment of random variable x_k, (1)
n	n-th delay, (2)
R_n	sample covariance at delay n, (2)
W_K	whiteness measure, (3)
Var	variance, (10)
V_K	variance of K-th whiteness measure W_K, (10)
A,B,C,D	unknown constants in (10) and (35)
ϕ_n	sum of delayed products of data, (11)
$A,\tilde{B},\tilde{C}$	constants for K odd, (45)
A,B,C	constants for K even, (48)
$p(x)$	probability density function of random variable x, (51)
D	constant for K even or odd, (55)
M	size of fast Fourier transform, (56)
$\{X_m\}$	fast Fourier transform of data $\{x_k\}$, (56)
u	threshold value, (56)
CDF(u)	cumulative distribution function, (57)
EDF(u)	exceedance distribution function, (57)
V_n	interval probabilities, (B-1)
C	cumulative distribution function, (B-3)
E	exceedance distribution function, (B-3)
μ_n	n-th moment, (B-8)

STATISTICS OF A WHITENESS MEASURE

INTRODUCTION

When a random number generator is designed to yield zero-mean independent random variables, one useful test of its validity is afforded by its sample covariance function. This quantity would ideally be zero for all delays except the origin value. However, in practice, due to the finite length of data generated and used to test the generator, the sample covariance function is not identically zero but fluctuates about zero. A measure of the whiteness of the generator is afforded by the sum of squares of all the off-zero elements of the sample covariance function, relative to the square of its origin value. This measure was suggested in [1; appendix C].

In this report, we investigate the statistics of this whiteness measure, including its cumulative and exceedance distribution functions and its mean and variance. Since a sample covariance involves products of data values, the squared covariance depends on fourth-order products of the data, and the variance of this sample quantity involves eighth-order products of the data under various delays. It is this latter high-order product which greatly complicates the statistical analysis and which necessitates a roundabout procedure for exact evaluation of the variance of the whiteness measure. The probability distributions of this measure are determined by simulation for two types of random variables, uniform and Gaussian.

MEAN AND VARIANCE OF WHITENESS MEASURE

Consider real data sequence $x_0, x_1, \ldots, x_{K-1}$ of K data points which are independent and identically distributed (IID) with a symmetric probability density function about zero. This zero-mean sequence will have all odd-order moments equal to zero. Also, assume that the data are scaled to have unit variance and a fourth moment of value F; that is

$$E\left(x_k^2\right) = 1 , \quad E\left(x_k^4\right) = F , \quad \text{for } 0 \le k \le K-1 , \tag{1}$$

where E denotes the expectation. This situation includes the uniform random number generator and the Gaussian random number generator, for example. For the usual uniform random variable distributed over $(-\frac{1}{2},\frac{1}{2})$, we have scaled its output by $\sqrt{12}$ for present purposes in order to realize variance 1. Thus, F = 1.8 for the uniform case, while F = 3 for Gaussian numbers.

The sample covariance of the available data is defined as

$$R_n = \frac{1}{K} \sum_k x_k \, x_{k-n} \quad \text{for all } n . \tag{2}$$

Ideally, we might like to have sequence $\{R_n\}$ equal to zero for $n \neq 0$. However, this is never the case, although the $\{R_n\}$ for $n \neq 0$ are much smaller than R_0 when K is large. The mean value of R_0 is easily seen to be 1, by reference to (1). A measure of the whiteness of data sequence $\{x_k\}$ is afforded by the sum of squares of all the off-zero elements of sequence $\{R_n\}$:

$$W_K \equiv \sum_{n\neq 0} R_n^2 = 2 \sum_{n=1}^{K-1} R_n^2 \quad \text{for } K \ge 2 . \tag{3}$$

MEAN OF WHITENESS MEASURE W_K

The mean value of random variable R_n^2 follows from (2) as

$$E\left(R_n^2\right) = E\left(\frac{1}{K^2} \sum_k \sum_j x_k \; x_{k-n} \; x_j \; x_{j-n}\right) =$$

$$= \frac{1}{K^2} \sum_k \sum_j E\left(x_k \; x_{k-n} \; x_j \; x_{j-n}\right) . \qquad (4)$$

Since we are only interested in values of $n > 0$ according to (3), the expectation in (4) is nonzero only when $k = j$; here, we are utilizing both the IID and the zero-mean properties of $\{x_k\}$. Then, (4) becomes, upon use of (1),

$$E\left(R_n^2\right) = \frac{1}{K^2} \sum_{k=n}^{K-1} 1 = \frac{K - n}{K^2} \quad \text{for } 1 \le n \le K-1 . \qquad (5)$$

(For completeness, $E\left(R_0^2\right) = (F + K - 1)/K$; Variance$(R_0) = (F-1)/K$. Thus, R_0 clusters around 1 as $K \to \infty$, while $R_n \to 0$ as $K \to \infty$ for fixed $n \ne 0$.) Use of result (5) in (3) yields the desired mean value of whiteness measure W_K as

$$E(W_K) = \frac{2}{K^2} \sum_{n=1}^{K-1} (K - n) = \frac{K - 1}{K} . \qquad (6)$$

Notice that this mean value is independent of fourth-moment F and that it approaches 1 as $K \to \infty$. Recall that $E(R_0) = 1$ for comparison.

VARIANCE OF WHITENESS MEASURE W_K

The direct evaluation of the variance of random variable W_K in (3) would require a very tedious procedure. Whereas the mean evaluation in (4) only encountered fourth-order products of delayed versions of $\{x_k\}$, we would now encounter eighth-order products, requiring a complicated counting procedure to account for all the various types of terms. Specifically, from (2) and (3), we have whiteness measure

$$W_K = \frac{2}{K^2} \sum_{n=1}^{K-1} \sum_{k=n}^{K-1} \sum_{j=n}^{K-1} x_k \, x_{k-n} \, x_j \, x_{j-n} \, , \tag{7}$$

leading to mean square value

$$E\left(W_K^2\right) = \frac{4}{K^4} \sum_{n=1}^{K-1} \sum_{m=1}^{K-1} \sum_{k=n}^{K-1} \sum_{j=n}^{K-1} \sum_{q=m}^{K-1} \sum_{p=m}^{K-1} E\left(x_k \, x_{k-n} \, x_j \, x_{j-n} \, x_q \, x_{q-m} \, x_p \, x_{p-m}\right) . \tag{8}$$

Not only would this eighth-order average have to be evaluated for all possible values of n,m,k,j,q,p, but the sixth-order summation would then have to be conducted. The only reasonable case that can be evaluated from (8) is that for the term proportional to F^2. It is obtained only for the special choices $n = m$ and $k = j = q = p$; then the right-hand side of (8) reduces to

$$\frac{4}{K^4} \sum_{n=1}^{K-1} \sum_{k=n}^{K-1} F^2 = \frac{4}{K^4} \sum_{n=1}^{K-1} (K - n) \, F^2 = \frac{2(K - 1)}{K^3} \, F^2 \, . \tag{9}$$

Notice that moments of $\{x_k\}$ above the fourth need not be known.

The difficulty of attempting to evaluate (8) directly forces us to attack the problem from a different aspect. Specifically, we adopt a shortcut to obtain, exactly, the variance of whiteness measure W_K. First, observe from (8) that the mean square value of W_K contains a denominator of K^4. Secondly, it has been observed from simulations that the variance of W_K goes to zero proportional to 1/K for large K. Therefore, the form of the variance, V_K, of random variable W_K must be

$$V_K = \mathrm{Var}(W_K) = \frac{A\,K^3 + B\,K^2 + C\,K + D}{K^4} \;, \tag{10}$$

where A, B, C, D are unknown constants. In order to determine these four constants, we will evaluate, exactly, the variance V_K of W_K for a sufficient number of low-order values of K, and then solve the four simultaneous linear equations yielded by (10).

For convenience, we define the sums

$$\phi_n = \sum_{k=n}^{K-1} x_k\, x_{k-n} \quad \text{for } 1 \le n \le K-1 \;. \tag{11}$$

Then

$$R_n = \frac{1}{K}\,\phi_n \quad \text{for } 1 \le n \le K-1 \;, \tag{12}$$

as seen from (2). The whiteness measure in (3) then takes the form

$$W_K = \frac{2}{K^2} \sum_{n=1}^{K-1} \phi_n^2 \quad \text{for } K \ge 2 \;. \tag{13}$$

For K = 1, there are no terms in the sum, yielding $W_1 = 0$.

SPECIAL CASE K = 2

We have, from (11) and (13),

$$\phi_1 = x_1 \; x_0 \;, \quad W_2 = \frac{2}{4} \; \phi_1^2 = \frac{1}{2} \; x_1^2 \; x_0^2 \;. \tag{14}$$

Therefore, upon use of the IID property of the $\{x_k\}$ and (1),

$$E\left(W_2^2\right) = \frac{1}{4} \; E\left(x_1^4 \; x_0^4\right) = \frac{1}{4} \; F^2 \;. \tag{15}$$

The variance of W_2 then follows as

$$V_2 = \mathrm{Var}(W_2) = E\left(W_2^2\right) - E\left(W_2\right)^2 = \frac{1}{4}\left(F^2 - 1\right) \;, \tag{16}$$

where we used (6).

SPECIAL CASE K = 3

The procedure for the remaining cases is similar to that detailed above for K = 2; therefore, the following presentation will be abbreviated, and only the main results will be listed. We have

$$\phi_1 = x_1 \; x_0 + x_2 \; x_1 \;, \quad \phi_2 = x_2 \; x_0 \;, \tag{17}$$

$$W_3 = \frac{2}{9}\left[\phi_1^2 + \phi_2^2\right] = \frac{2}{9}\left[x_1^2 (x_0 + x_2)^2 + x_2^2 \; x_0^2\right] \;, \tag{18}$$

$$W_3^2 = \frac{4}{81}\left[x_1^4 (x_0 + x_2)^4 + x_2^4 \; x_0^4 + 2 \; x_1^2 (x_0 + x_2)^2 \; x_2^2 \; x_0^2\right] \;. \tag{19}$$

The mean value of (19) is given by

$$E\left(W_3^2\right) = \frac{4}{81}\left(F(F + 6 + F) + F^2 + 2(F + F)\right) = \frac{4}{81}\left(3F^2 + 10F\right) . \quad (20)$$

Finally, the variance of W_3 is

$$V_3 = \frac{4}{81}\left(3F^2 + 10F - 9\right) . \quad (21)$$

SPECIAL CASE K = 4

In this case, we have

$$\phi_1 = x_1 x_0 + x_2 x_1 + x_3 x_2 , \quad \phi_2 = x_2 x_0 + x_3 x_1 , \quad \phi_3 = x_3 x_0 , \quad (22)$$

$$W_4 = \frac{2}{16}\left[\phi_1^2 + \phi_2^2 + \phi_3^2\right] , \quad (23)$$

$$64 W_4^2 = \phi_1^4 + \phi_2^4 + \phi_3^4 + 2 \phi_1^2 \phi_2^2 + 2 \phi_1^2 \phi_3^2 + 2 \phi_2^2 \phi_3^2 . \quad (24)$$

The mean value of (24) will be found in stages. The six components of (24) have the following average values:

$$E\left(\phi_3^4\right) = E\left(x_3^4 x_0^4\right) = F^2 , \quad (25)$$

$$E\left(\phi_3^2 \phi_2^2\right) = E\left(x_3^2 x_0^2(x_2 x_0 + x_3 x_1)^2\right) = F + F = 2F , \quad (26)$$

$$E\left(\phi_3^2 \phi_1^2\right) = E\left(x_3^2 x_0^2(x_1 x_0 + x_2 x_1 + x_3 x_2)^2\right) = F{+}1{+}F = 2F + 1 , \quad (27)$$

$$E\left(\phi_2^4\right) = E\left((x_2 x_0 + x_3 x_1)^4\right) = F^2 + 6 + F^2 = 2F^2 + 6 , \quad (28)$$

$$E\left(\phi_2^2\ \phi_1^2\right) = E\left((x_2\ x_0 + x_3\ x_1)^2\ (x_1\ x_0 + x_2\ x_1 + x_3\ x_2)^2\right) =$$

$$= E\left(\left[x_2^2\ x_0^2 + x_3^2\ x_1^2 + 2\ x_3\ x_2\ x_1\ x_0\right]\left[x_1^2\ x_0^2 + x_2^2\ x_1^2 +\right.\right.$$

$$\left.\left. + x_3^2\ x_2^2 + 2\ x_2\ x_1^2\ x_0 + 2\ x_3\ x_2\ x_1\ x_0 + 2\ x_3\ x_2^2\ x_1\right]\right) =$$

$$= F + F + F + F + F + F + 4 = 6F + 4\ , \tag{29}$$

$$\phi_1^2 = x_1^2(x_0 + x_2)^2 + x_3^2\ x_2^2 + 2\ x_3\ x_2\ x_1(x_0 + x_2)\ , \tag{30}$$

$$\phi_1^4 = x_1^4(x_0 + x_2)^4 + x_3^4\ x_2^4 + 6\ x_3^2\ x_2^2\ x_1^2(x_0 + x_2)^2 +$$

$$+ 4\ x_3\ x_2\ x_1^3(x_0 + x_2)^3 + 4\ x_3^3\ x_2^3\ x_1(x_0 + x_2)\ , \tag{31}$$

$$E\left(\phi_1^4\right) = F(F + 6 + F) + F^2 + 6(1 + F) = 3F^2 + 12F + 6\ . \tag{32}$$

Combining these results into (24), we have mean square value

$$E\left(W_4^2\right) = \frac{1}{32}\left(3F^2 + 16F + 11\right) \tag{33}$$

and variance

$$V_4 = \frac{1}{32}\left(3F^2 + 16F - 7\right)\ . \tag{34}$$

The analytical derivations of V_5, V_6, V_7, V_8 are deferred to appendix A due to their lengthy calculations and need for a shorthand notation. It will turn out that we also need all of these latter results when we find the constants A, B, C, D in variance expression (10).

GENERAL DETERMINATION OF VARIANCE OF W_K

The general form for the variance V_K of whiteness measure W_K is given by (10) for arbitrary K and is repeated below:

$$V_K = \mathrm{Var}(W_K) = \frac{A\,K^3 + B\,K^2 + C\,K + D}{K^4} . \qquad (35)$$

However, analytic determination of V_K for K = 2, 3, 4, 5, 6, 7, 8 (see appendix A also) have revealed that separate forms like (35) must be employed for K even versus K odd. That is, two <u>different</u> sets of constants A, B, C, D apply in the even versus odd cases of K. The available analytic results for V_K (above and in appendix A) are summarized below:

$$V_1 = 0 \quad \text{(see the line under (13))} , \qquad (36)$$

$$V_2 = \frac{1}{4}\left(F^2 - 1\right) , \qquad (37)$$

$$V_3 = \frac{4}{81}\left(3F^2 + 10F - 9\right) , \qquad (38)$$

$$V_4 = \frac{1}{32}\left(3F^2 + 16F - 7\right) , \qquad (39)$$

$$V_5 = \frac{8}{625}\left(5F^2 + 38F - 11\right) , \qquad (40)$$

$$V_6 = \frac{1}{324}\left(15F^2 + 144F - 23\right) , \qquad (41)$$

$$V_7 = \frac{4}{2401}\left(21F^2 + 246F - 23\right) , \qquad (42)$$

$$V_8 = \frac{1}{256}\left(7F^2 + 96F - 3\right) . \qquad (43)$$

If we take K equal to the odd values 1, 3, 5, 7 in (35) and use results (36), (38), (40), (42), we obtain four simultaneous linear equations for the constants A, B, C, D. Their solution leads to the following expression for the variance V_K of W_K:

$$V_K = \frac{K-1}{K^4}\left(A\ K^2 + \tilde{B}\ K + \tilde{C}\right) \quad \text{for K odd ,} \tag{44}$$

where

$$A = 4F + \frac{4}{3}\ , \quad \tilde{B} = 2F^2 - 4F - \frac{38}{3}\ , \quad \tilde{C} = -\ 4F + 8\ . \tag{45}$$

When (45) is substituted into (44), the variance expression can be rearranged in terms of powers of F:

$$V_K = \frac{2(K-1)}{K^4}\left[KF^2 + 2\left(K^2-K-1\right)F + \frac{1}{3}\left(2K^2-19K+12\right)\right] \quad \text{for K odd .} \tag{46}$$

The F^2 term here confirms (9), as anticipated.

If we take K equal to the even values 2, 4, 6, 8 in (35) and use results (37), (39), (41), (43), we obtain four different simultaneous linear equations for the constants A, B, C, D. Their solution leads to the following expression for the variance V_K of W_K:

$$V_K = \frac{1}{K^3}\left(A\ K^2 + B\ K + C\right) \quad \text{for K even ,} \tag{47}$$

where

$$A = 4F + \frac{4}{3}\ , \quad B = 2F^2 - 8F - 14\ , \quad C = -\ 2F^2 + \frac{62}{3}\ . \tag{48}$$

When (48) is substituted into (47), the variance expression can be rearranged in terms of powers of F according to

$$V_K = \frac{2}{K^3}\left[(K-1)F^2 + 2K(K-2)F + \frac{1}{3}\left(2K^2-21K+31\right)\right] \quad \text{for K even} . \quad (49)$$

Again, the F^2 dependence in (9) is confirmed.

The asymptotic behavior of variance V_K for large K is given by

$$V_K \sim \left(4F + \frac{4}{3}\right) \frac{1}{K} \quad \text{as } K \rightarrow \infty \quad (50)$$

for both K odd and K even. This is due to the fact that constant A in (35) is identical for the odd and even cases; compare (45) and (48). Thus, whiteness measure W_K tends to cluster around 1 as $K \rightarrow \infty$. Recall that $R_0 \rightarrow 1$, while $R_n \rightarrow 0$ for fixed n, as $K \rightarrow \infty$.

The end results for variance V_K of whiteness measure W_K are given by (44) and (47), or by (46) and (49). Plots of V_K for the uniform random variable and the Gaussian random variable $\{x_k\}$ are displayed in figures 1 and 2, respectively. A short tabulation of V_K is given in table 1 for the uniform, Gaussian, exponential, and alternating random variables $\{x_k\}$. The probability density functions of $\{x_k\}$ for these four cases are, respectively,

$$p_u(x) = .5/\sqrt{3} \quad \text{for } |x| < \sqrt{3} , \quad F = 1.8 ; \quad (51)$$

$$p_g(x) = (2\pi)^{-\frac{1}{2}} \exp(-x^2/2) , \quad F = 3 ; \quad (52)$$

$$p_e(x) = \frac{1}{\sqrt{2}} \exp(-\sqrt{2}|x|) , \quad F = 6 ; \quad (53)$$

$$p_a(x) = \frac{1}{2}\, \delta(x-1) + \frac{1}{2}\, \delta(x+1) , \quad F = 1 . \quad (54)$$

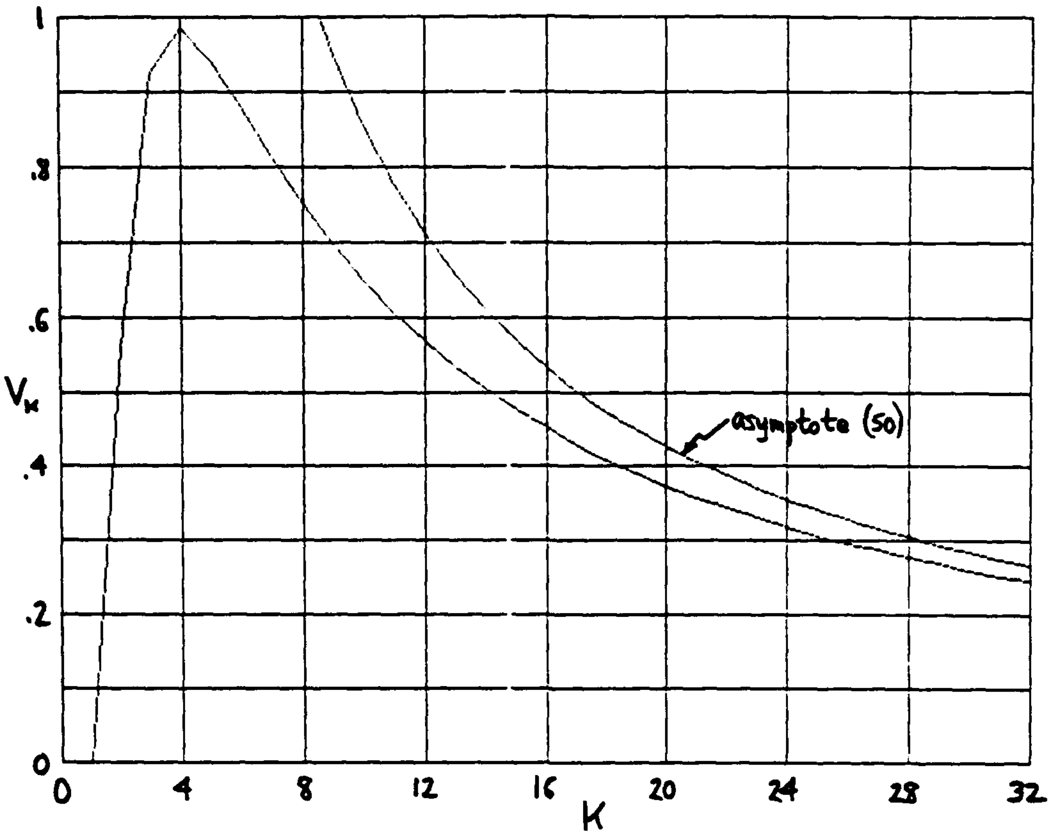

Figure 1. Variance V_K for Uniform Random Variables $\{x_k\}$

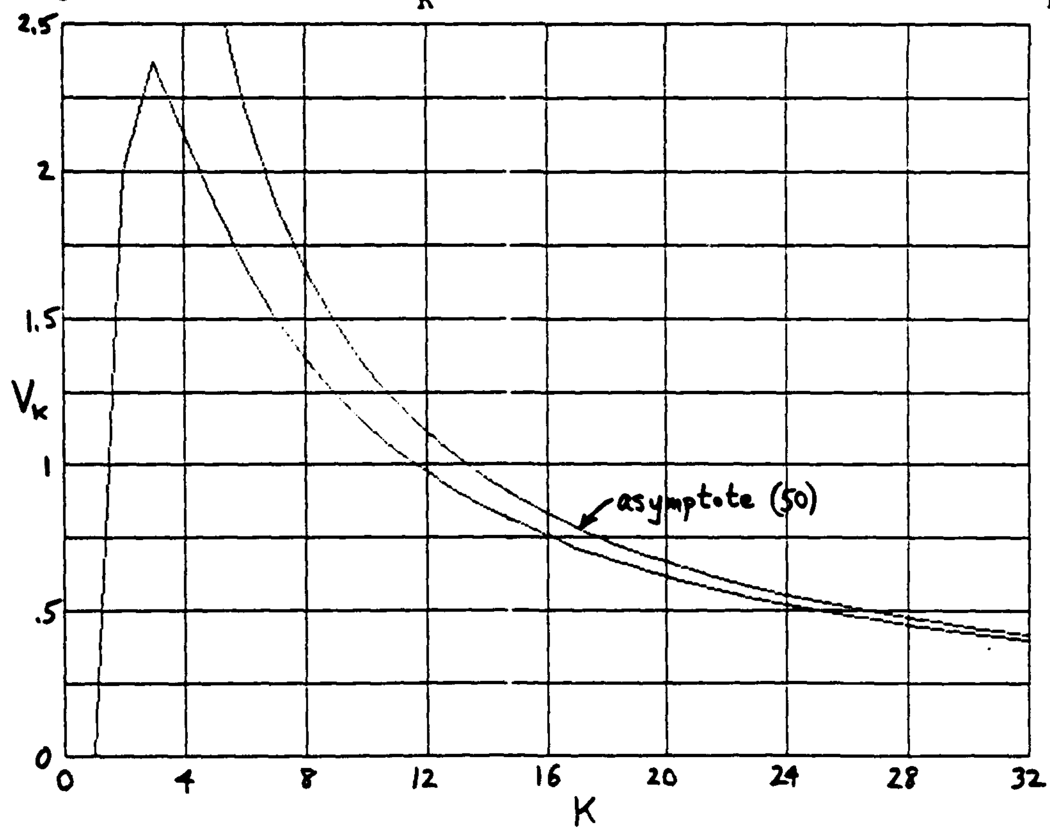

Figure 2. Variance V_K for Gaussian Random Variables $\{x_k\}$

A short table of the variances for these four examples is given below. For the alternating example, $x_k = \pm 1$ and $F = 1$, whiteness measure W_2 for $K = 2$ is always equal to 1/2, thereby leading to variance $V_2 = 0$. The smallest possible example of F is 1, as realized in the alternating random variable case.

Table 1. Variance V_K of Whiteness Measure W_K

K	Uniform	Gaussian	Exponential	Alternating
2	.56	2.	8.75	.0
3	.92444	2.37037	7.85185	.19753
4	.985	2.125	6.15625	.375
5	.94208	1.8944	5.0816	.4096
6	.87901	1.67901	4.26235	.41975
7	.81273	1.50604	3.68013	.40650
8	.75188	1.35938	3.22266	.39063
16	.45117	.75586	1.60986	.25977
32	.24569	.39722	.79987	.14771
64	.12804	.20346	.39808	.07852
128	.06534	.10295	.19850	.04045

If we combine (47) with the multiplied-out version of (44), the variance V_K can indeed be written in the form (35) for <u>all</u> K, where the constants A, B, C are as given in (48), but constant D must be taken according to the two different values

$$D = \begin{Bmatrix} 0 & \text{for } K \text{ even} \\ 4(F-2) & \text{for } K \text{ odd} \end{Bmatrix} . \tag{55}$$

Notice that, despite (8) involving eighth-order products, nothing above fourth-order moment F of $\{x_k\}$ is required in these results.

PROBABILITY DISTRIBUTIONS OF WHITENESS MEASURE

The direct evaluation of whiteness measure W_K, according to its definition (3) in conjunction with (2), is very time consuming for large K, due to all the multiplications required. An attractive alternative, in terms of fast Fourier transforms, was derived in [1; appendix C] and is employed here; the program utilized is listed in appendix B. The key relation relative to (3) is [1; (C-5)]

$$W_K = \frac{1}{K^2 M^2} \left[M \sum_{m=0}^{M-1} |X_m|^4 - \left(\sum_{m=0}^{M-1} |X_m|^2 \right)^2 \right] , \tag{56}$$

where M is the size of the fast Fourier transform $\{X_m\}$ of data $\{x_k\}$. The only restriction on M is that we must use $M \ge 2K - 1$; then, the right-hand side of (56) is independent of M. (For $K = 1$, $X_m = x_0$ for $0 \le m \le M-1$, leading to $W_1 = 0$, as noted under (13).) Again, notice that the whiteness measure W_K depends on fourth-order products of the data or its transform.

The cumulative distribution function (CDF) and exceedance distribution function (EDF) of whiteness measure W_K,

$$\mathrm{CDF}(u) = \mathrm{Prob}(W_K < u) , \quad \mathrm{EDF}(u) = \mathrm{Prob}(W_K > u) , \tag{57}$$

for the case where data $\{x_k\}$ is uniformly distributed over $-\sqrt{3},\sqrt{3}$ [see (51)], are displayed in figures 3 - 10 for K = 2, 3, 4, 8, 16, 32, 64, 128, respectively. These results were determined by using at least one million trials for W_K as defined in (56). The

exceedance distribution function for small K has a cusp near zero argument which disappears for larger K. However, random variable W_K does not approach Gaussian as K increases; rather, as shown in figure 10 for K = 128, the right-hand tail appears to approach exponential behavior. For a bounded random variable, $|x_k| < B$, the value of W_K is bounded according to

$$W_K < \frac{(K - 1)(2K - 1)}{3K} B^4 . \qquad (58)$$

In the case of the uniform random variable x_k, where $B = \sqrt{3}$, (58) yields 4.5 for K = 2, 10 for K = 3, and 15.75 for K = 4.

Although the mean of W_{128} is 127/128 and its variance is $V_{128} = .06534$, the standard deviation of W_{128} is 0.256; this leads to the possibility of large values of W_{128} on occasion. For example, figure 10 shows that the whiteness measure can reach a value of 1.8 or larger about 1% of the time. If a candidate uniform random number generator has probability distributions for W_K which differ significantly from figures 3 - 10, it is suspect and should be more thoroughly investigated before further use.

The corresponding cumulative and exceedance distribution functions of the whiteness measure W_K for a Gaussian random number generator [see (52)] are displayed in figures 11 - 18 for K = 2, 3, 4, 8, 16, 32, 64, 128, respectively. The first observation to make is that the positive tail of W_K can now reach much larger values when K is small. However, for the larger values of K, the probability distributions of W_K appear to be approaching a common behavior, regardless of the distribution of the underlying data $\{x_k\}$; compare figures 10 and 18 for K = 128.

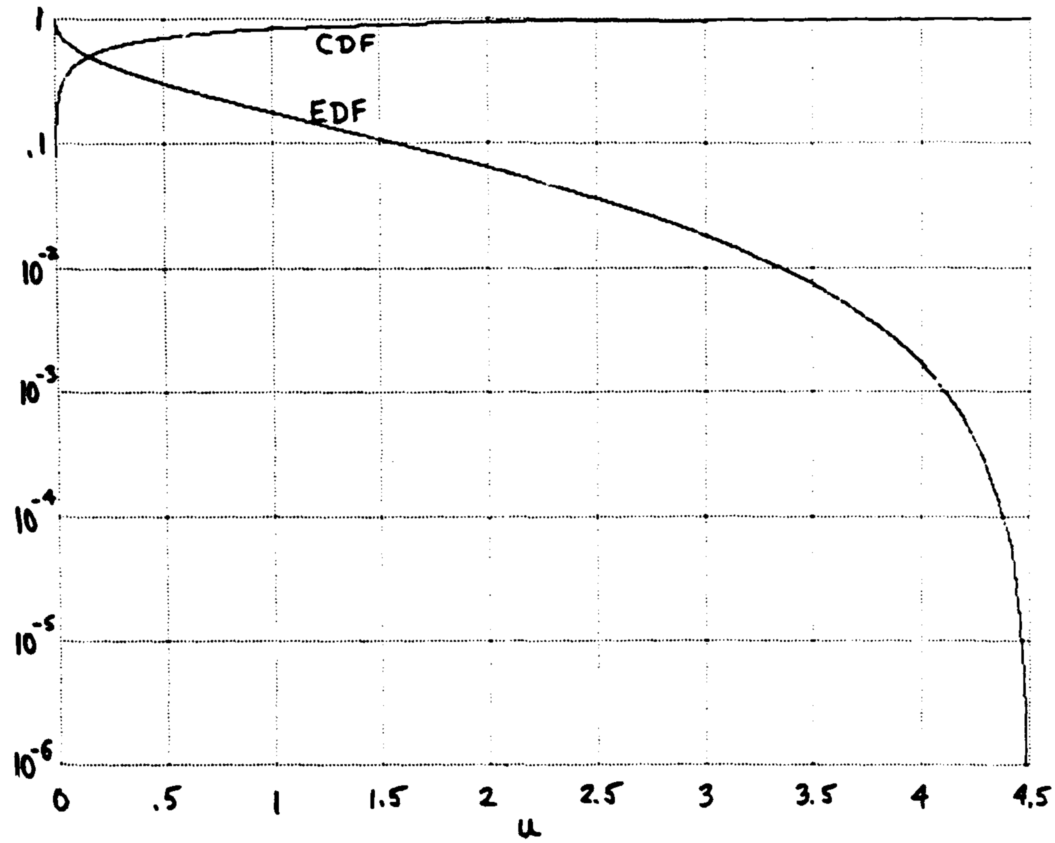

Figure 3. Distributions of W_2 for Uniform Random Variables

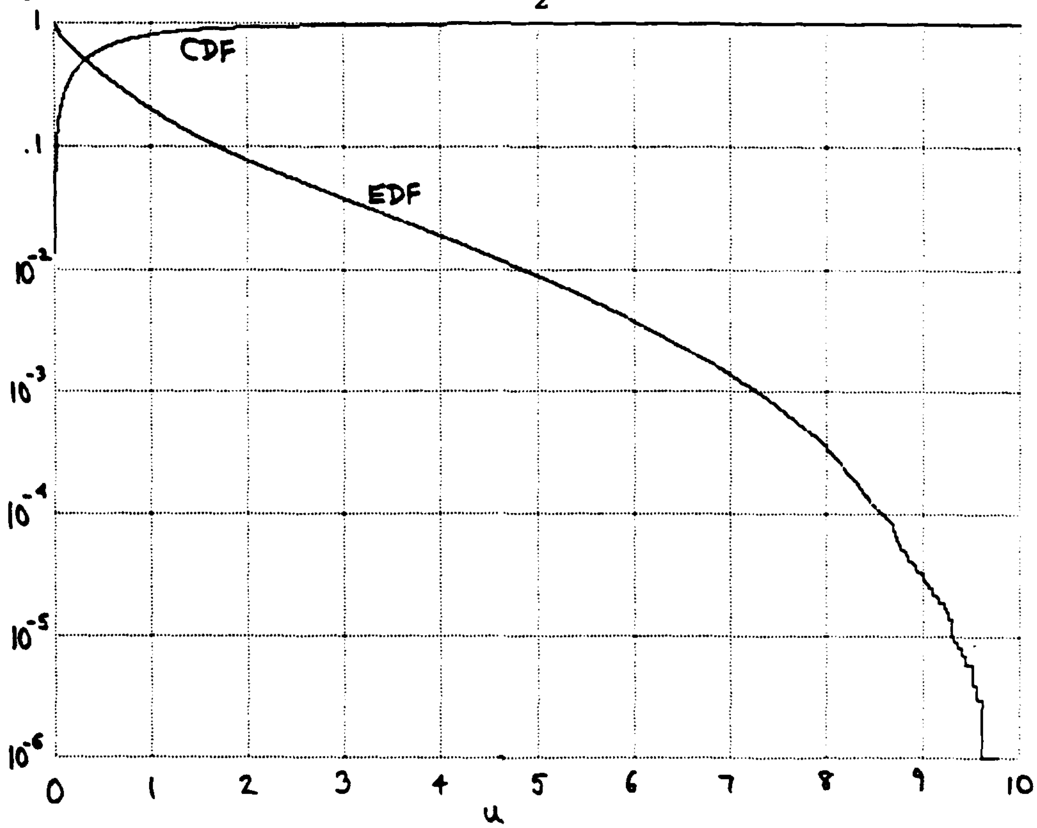

Figure 4. Distributions of W_3 for Uniform Random Variables

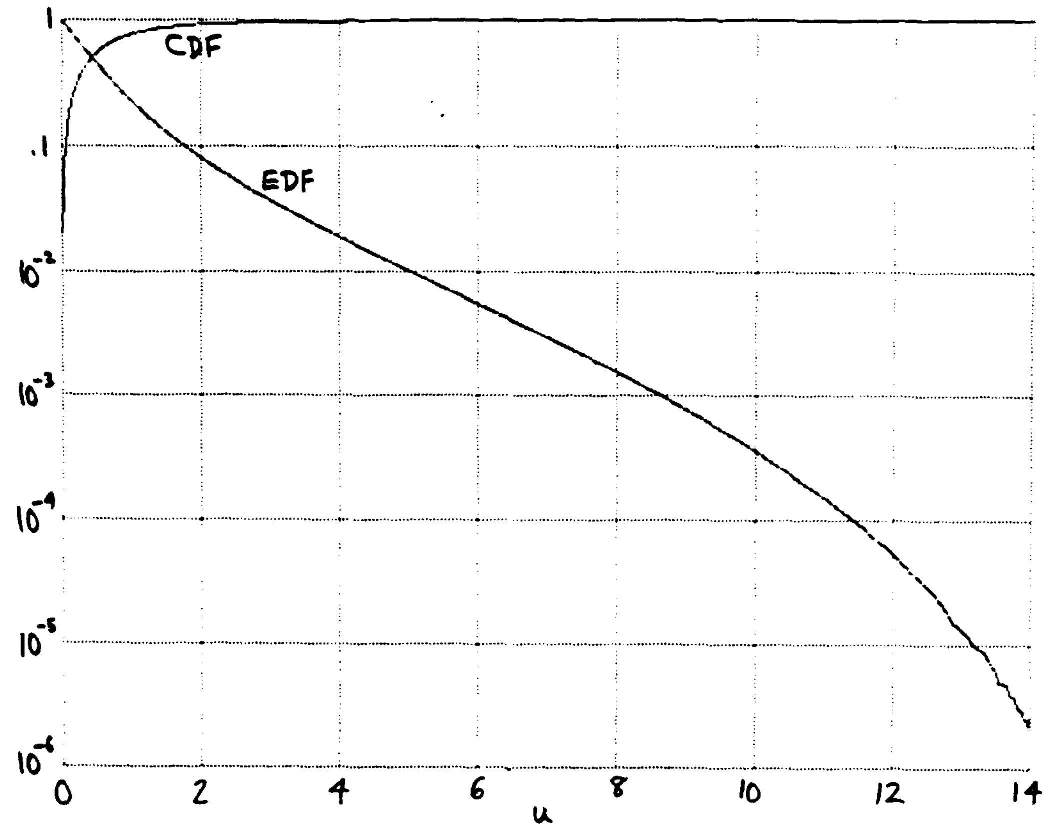

Figure 5. Distributions of W_4 for Uniform Random Variables

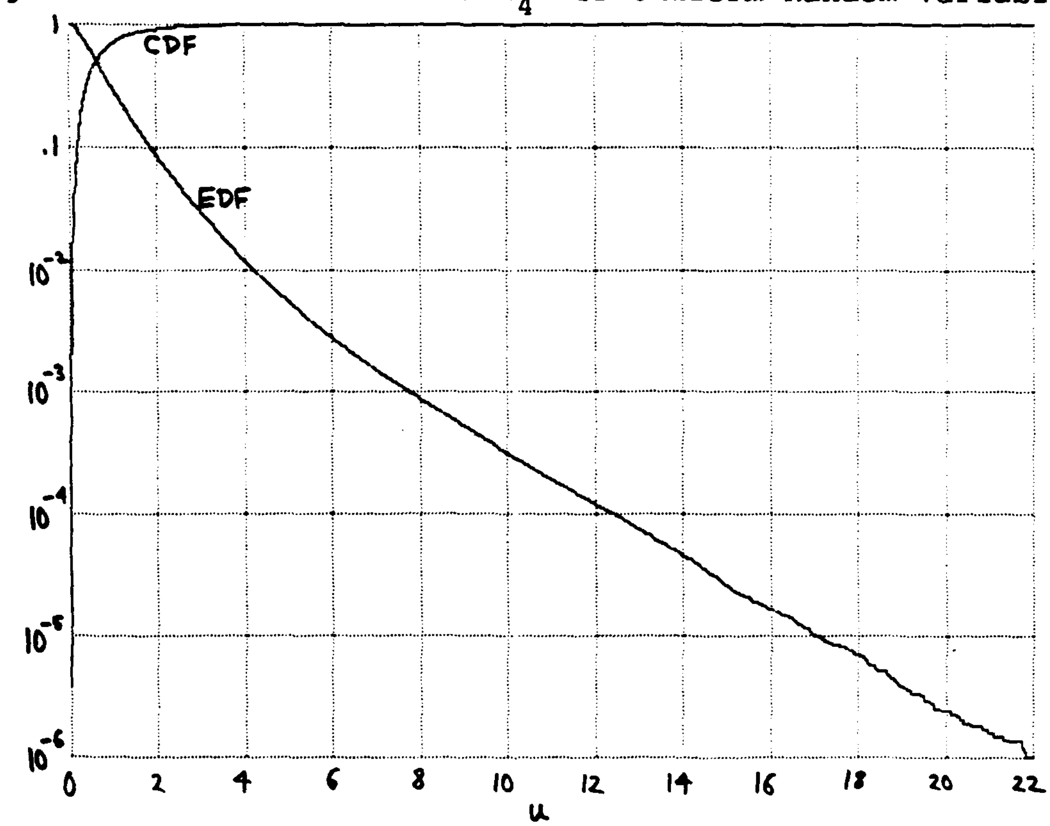

Figure 6. Distributions of W_8 for Uniform Random Variables

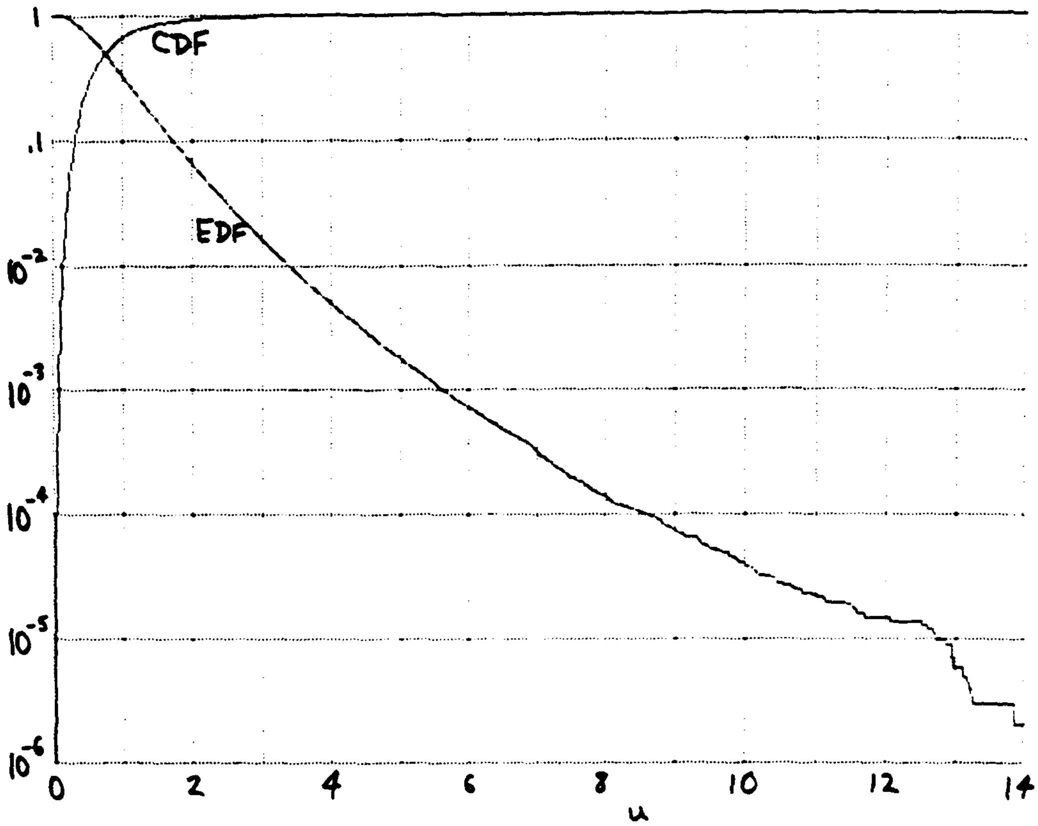

Figure 7. Distributions of W_{16} for Uniform Random Variables

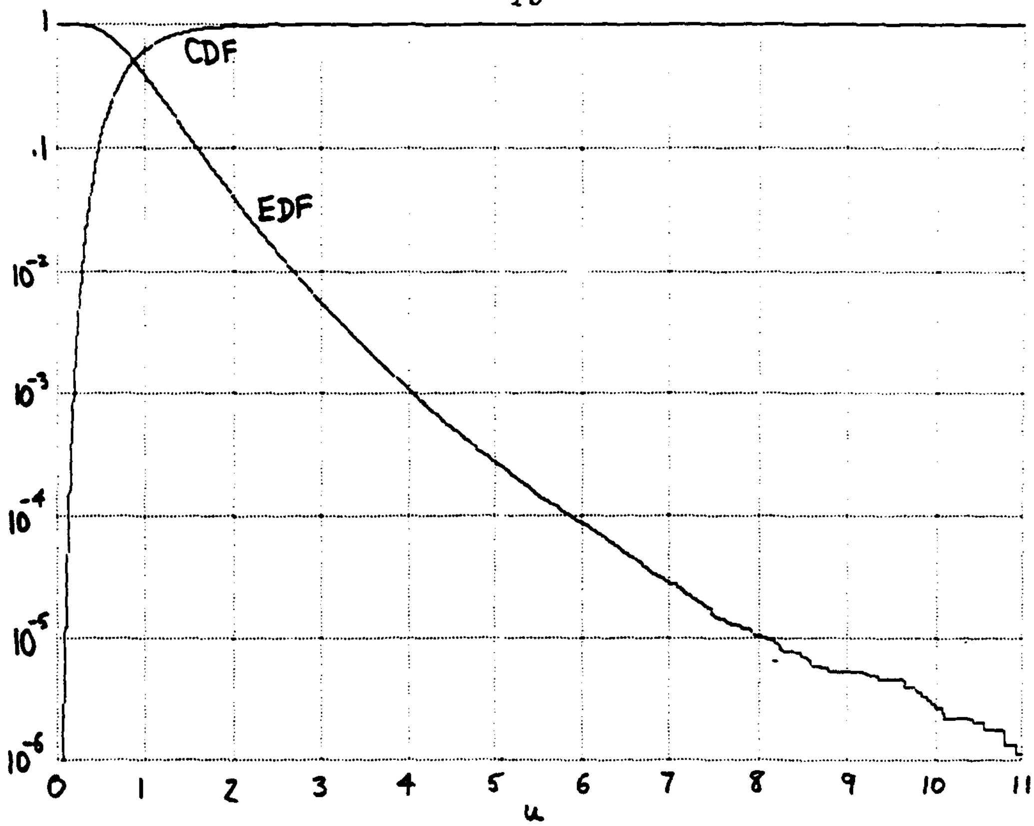

Figure 8. Distributions of W_{32} for Uniform Random Variables

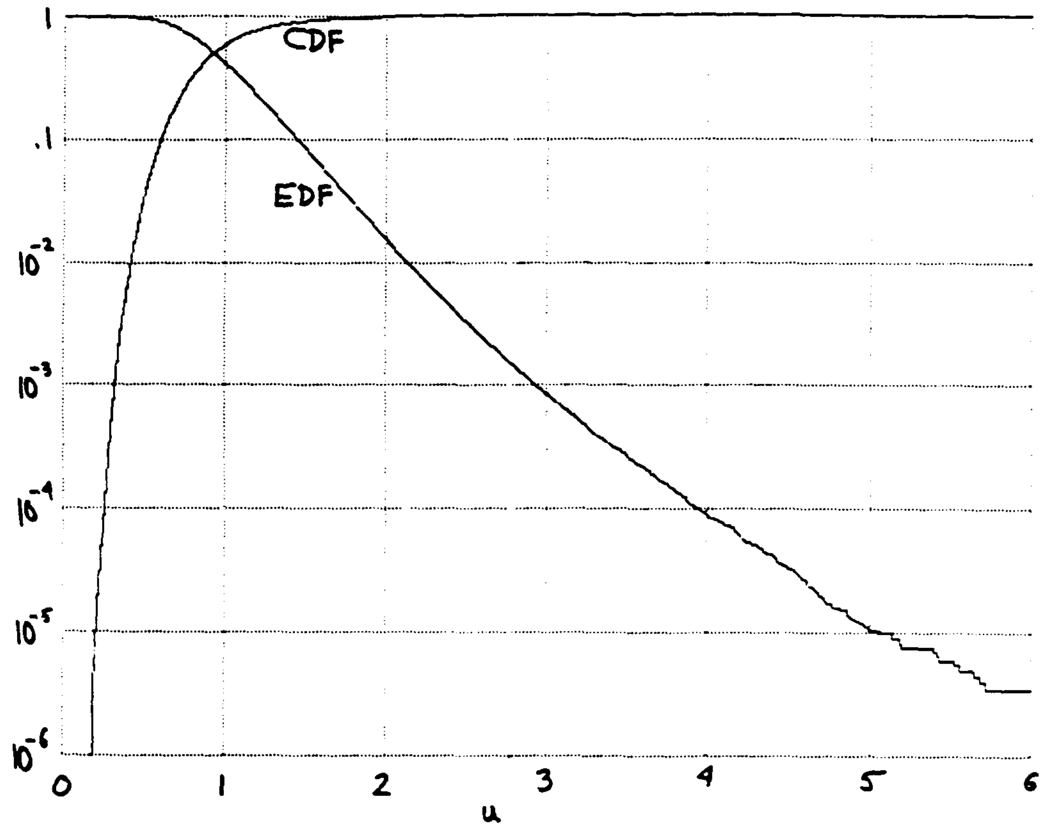

Figure 9. Distributions of W_{64} for Uniform Random Variables

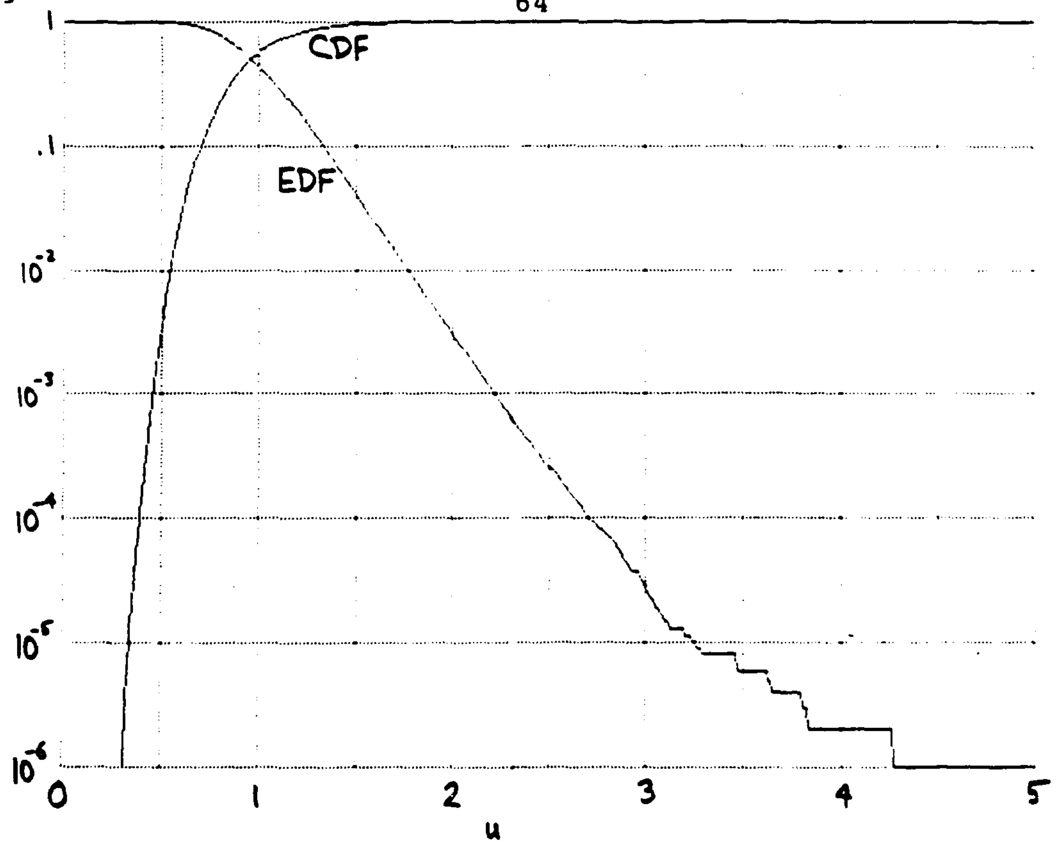

Figure 10. Distributions of W_{128} for Uniform Random Variables

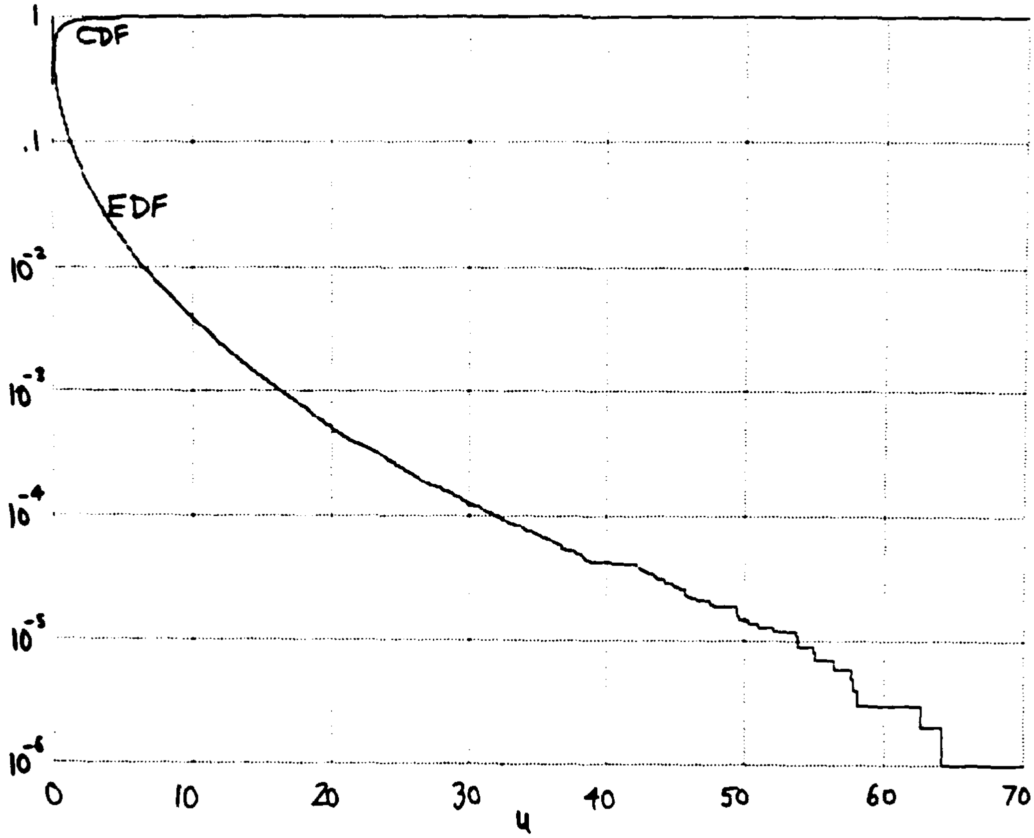

Figure 11. Distributions of W_2 for Gaussian Random Variables

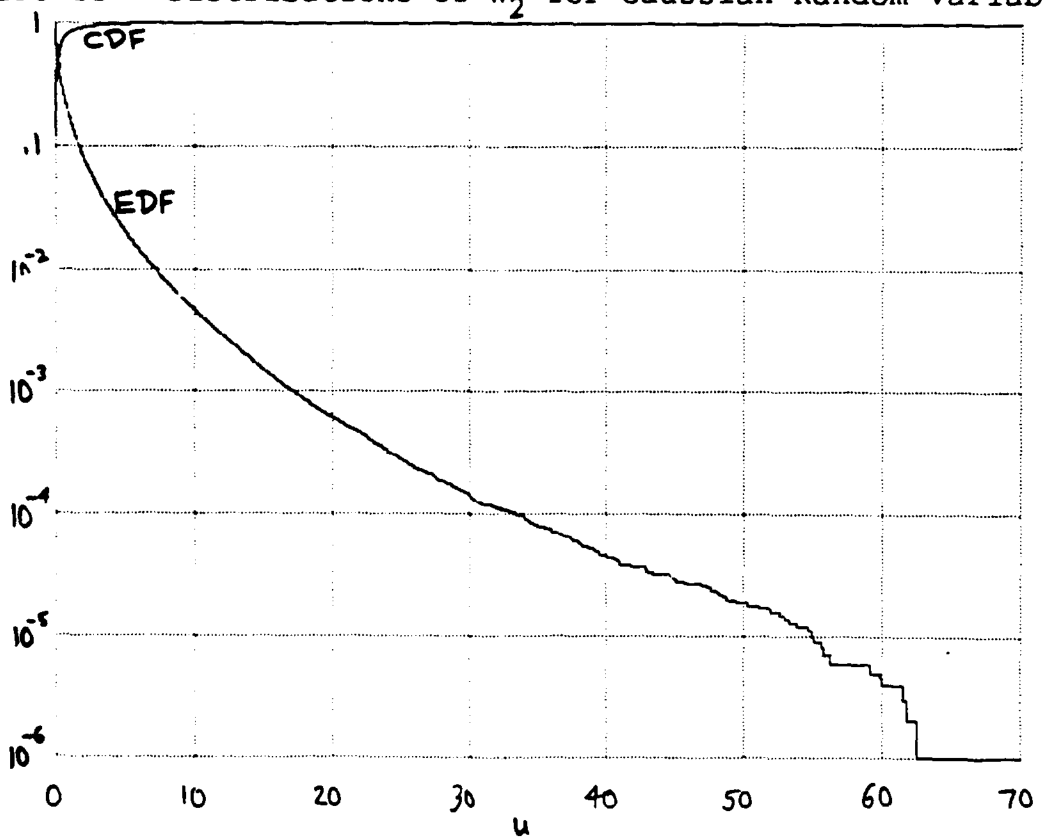

Figure 12. Distributions of W_3 for Gaussian Random Variables

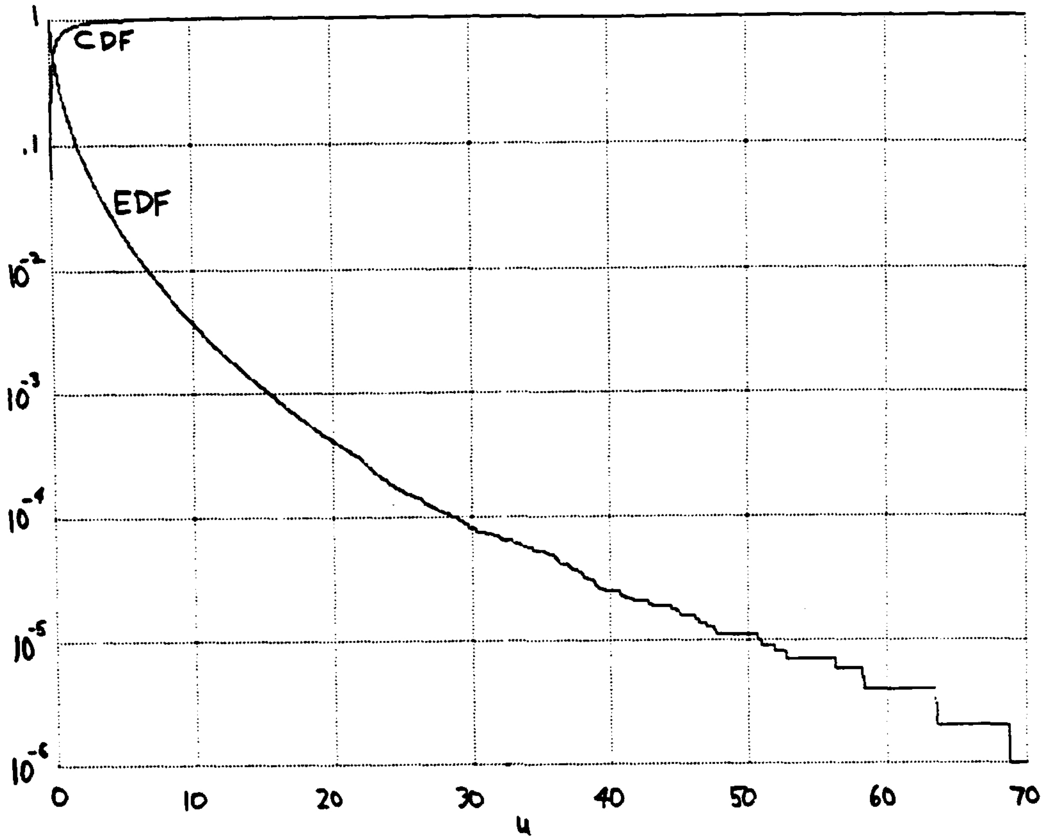

Figure 13. Distributions of W_4 for Gaussian Random Variables

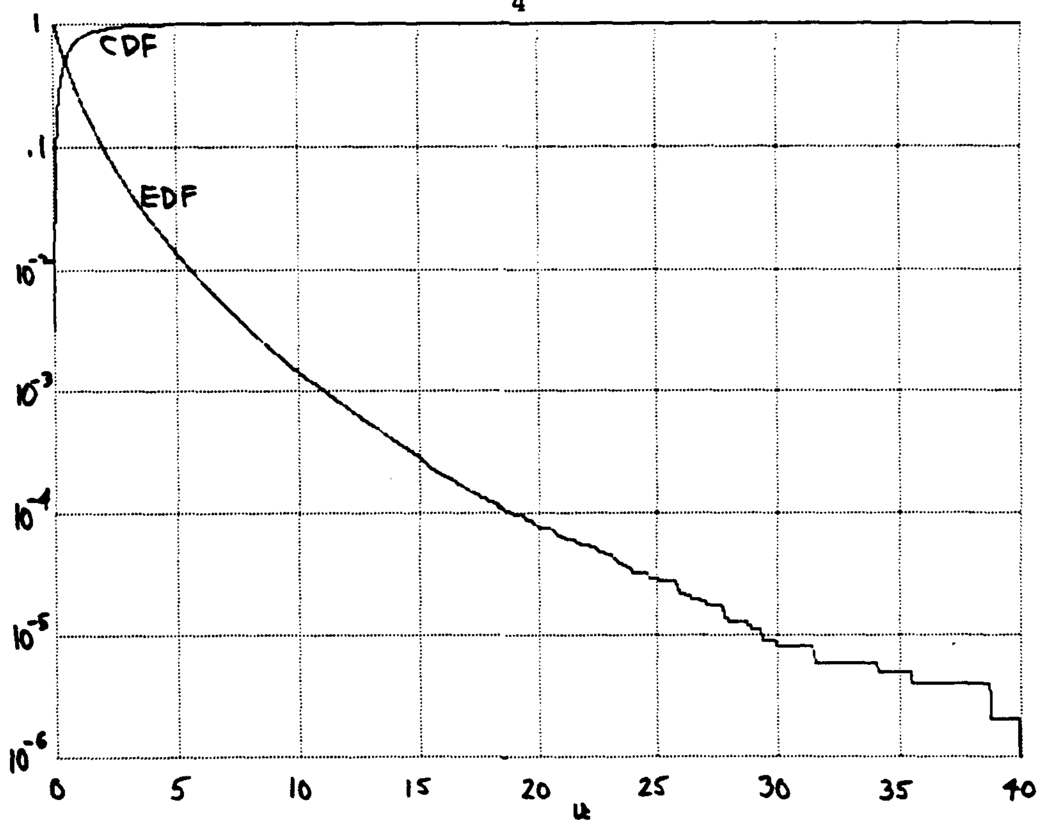

Figure 14. Distributions of W_8 for Gaussian Random Variables

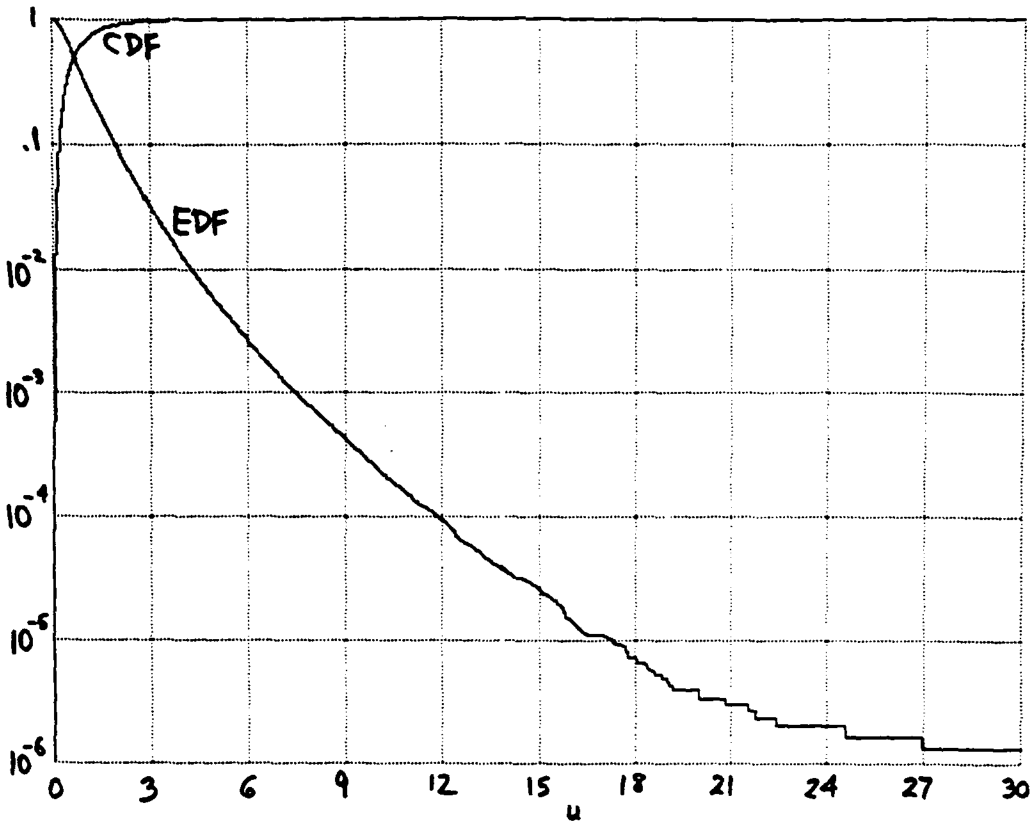

Figure 15. Distributions of W_{16} for Gaussian Random Variables

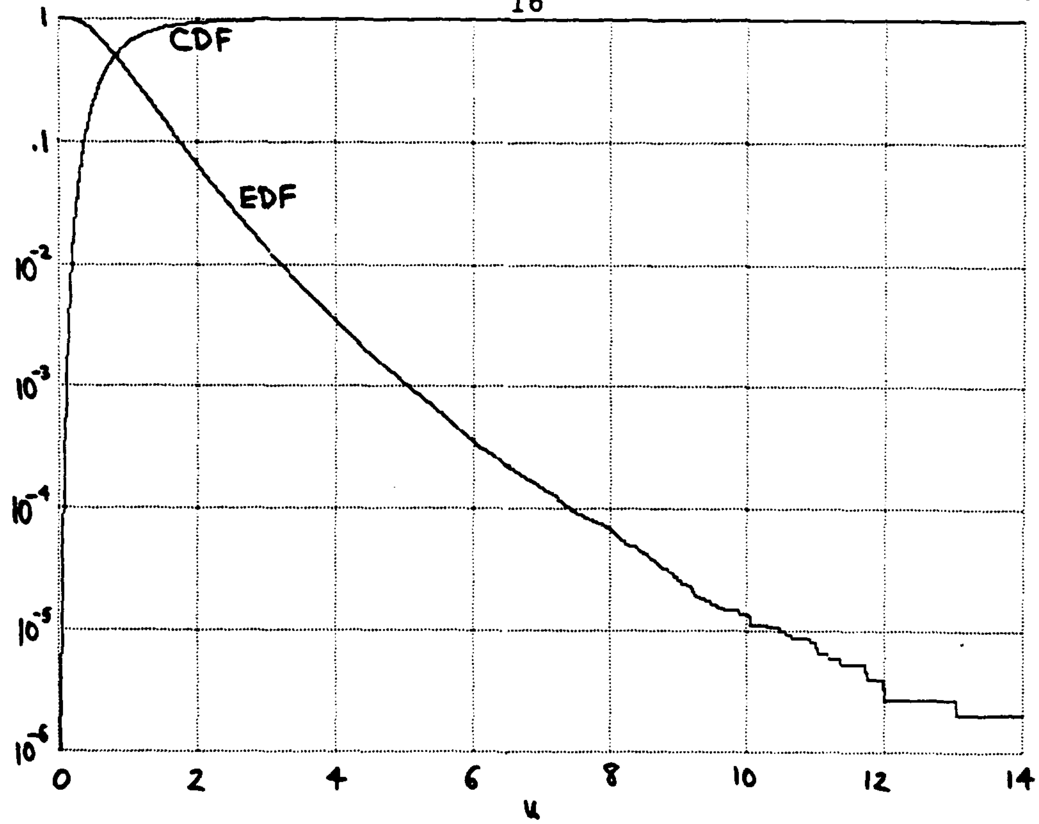

Figure 16. Distributions of W_{32} for Gaussian Random Variables

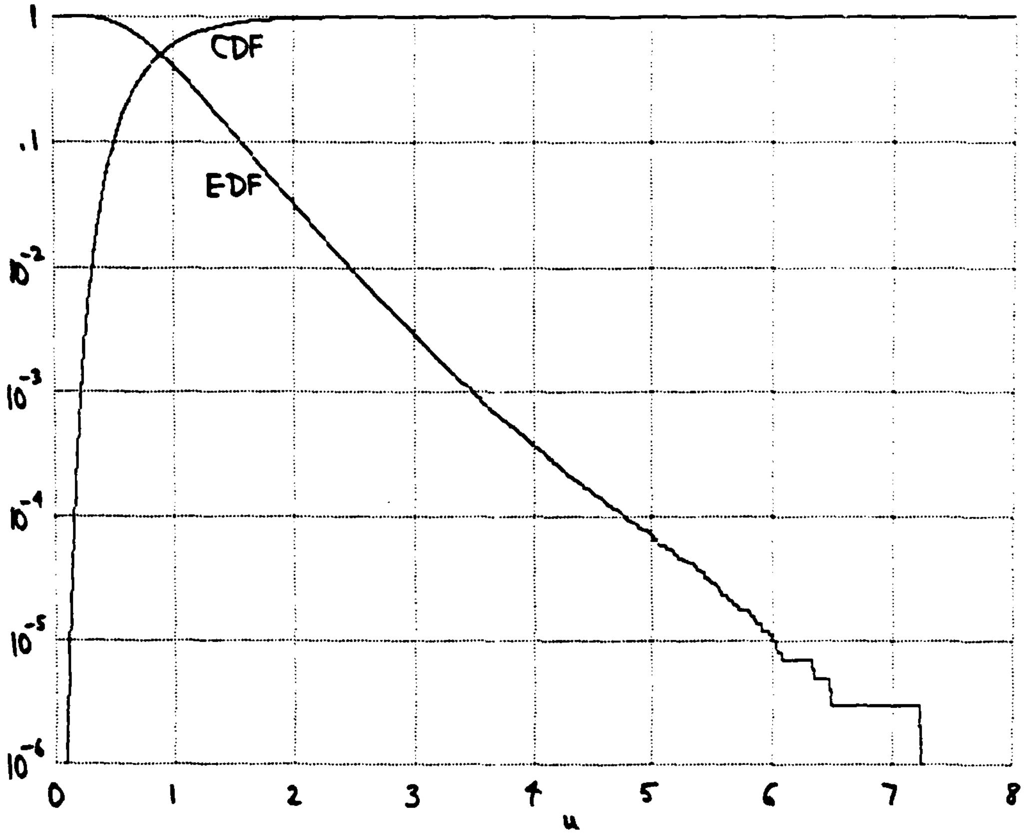

Figure 17. Distributions of W_{64} for Gaussian Random Variables

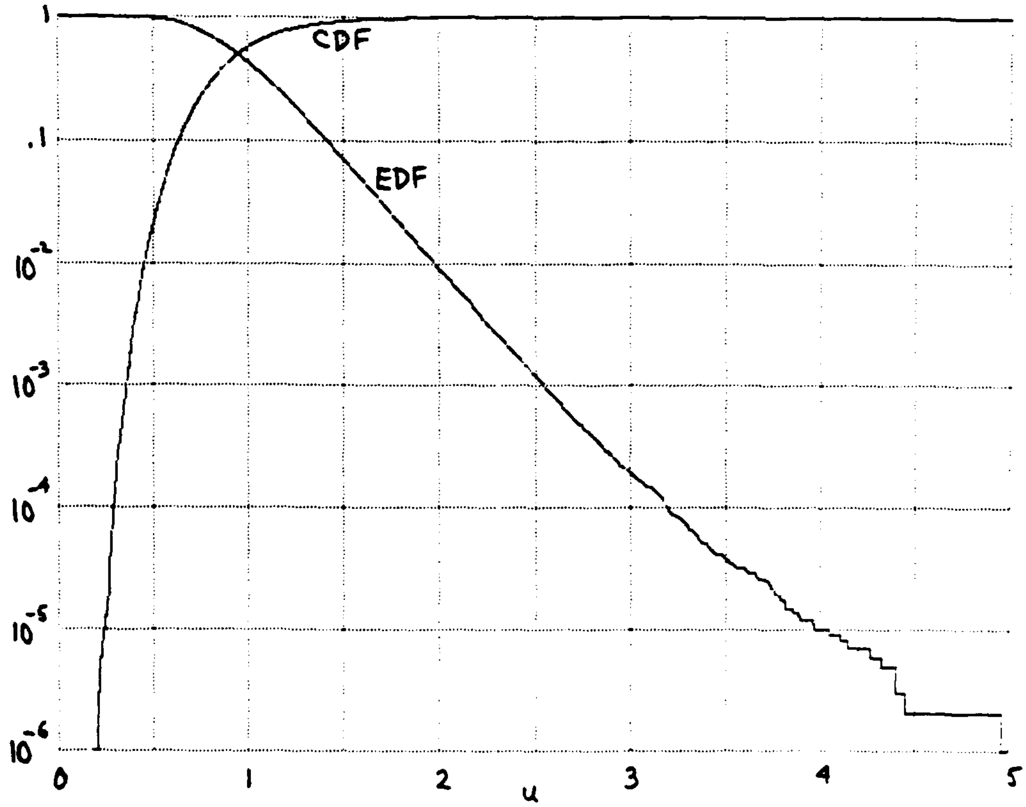

Figure 18. Distributions of W_{128} for Gaussian Random Variables

SUMMARY

The statistics of a whiteness measure, for testing a random number generator, have been investigated in terms of the mean, variance, and probability distributions. The mean and variance results are exact and have been borne out by numerous simulations for different noise sources $\{x_k\}$ and data sizes K. These results, for whiteness measure W_K defined in (3), are summarized below:

$$E(W_K) = \frac{K-1}{K}\,, \quad V_K = \mathrm{Var}(W_K) = \frac{A\,K^3 + B\,K^2 + C\,K + D}{K^4}\,, \tag{59}$$

where

$$A = 4F + \frac{4}{3}\,, \quad B = 2F^2 - 8F - 14\,, \quad C = -\,2F^2 + \frac{62}{3} \quad \text{for all } K\,,$$

$$\text{while} \quad D = \left\{ \begin{array}{ll} 0 & \text{for } K \text{ even} \\ 4(F-2) & \text{for } K \text{ odd} \end{array} \right\}. \tag{60}$$

The mean of whiteness measure W_K is independent of fourth-order moment F, while the variance of W_K depends on F, but not on sixth or eighth-order moments of data $\{x_k\}$. That is, the eighth-order product encountered in the general mean-square expression (8) never requires knowledge higher than fourth-order for its evaluation. This result applies for a symmetric zero-mean probability density function for unit-variance data $\{x_k\}$.

The cumulative and exceedance probability distributions were determined by simulations involving more than one million trials each and therefore have good reliability approximately down to the .0001 probability level.

APPENDIX A. DERIVATION OF VARIANCE OF WHITENESS MEASURE W_K

The variances V_K of whiteness measure W_K for $K = 2, 3, 4$ were derived in (14) - (34) in the main text. We now present the derivations for the remaining cases, $K = 5, 6, 7, 8$, that are necessary in order to determine V_K for all K.

SPECIAL CASE K = 5

$$\phi_1 = x_1\, x_0 + x_2\, x_1 + x_3\, x_2 + x_4\, x_3 \,, \qquad \phi_4 = x_4\, x_0 \,,$$

$$\phi_2 = x_2\, x_0 + x_3\, x_1 + x_4\, x_2 \,, \quad \phi_3 = x_3\, x_0 + x_4\, x_1 \,, \tag{A-1}$$

$$W_5 = \frac{2}{25}\left[\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right] , \tag{A-2}$$

$$\frac{625}{4}\, W_5^2 = \phi_1^4 + \phi_2^4 + \phi_3^4 + \phi_4^4 + 2\,\phi_1^2\,\phi_2^2 + 2\,\phi_1^2\,\phi_3^2 + 2\,\phi_1^2\,\phi_4^2 +$$

$$+ 2\,\phi_2^2\,\phi_3^2 + 2\,\phi_2^2\,\phi_4^2 + 2\,\phi_3^2\,\phi_4^2 \,. \tag{A-3}$$

The component averages required are developed in detail as follows:

$$E\left(\phi_4^4\right) = E\left(x_4^4\; x_0^4\right) = F^2 \,, \tag{A-4}$$

$$E\left(\phi_4^2\;\phi_3^2\right) = E\left(x_4^2\; x_0^2 (x_3\, x_0 + x_4\, x_1)^2\right) = F + F = 2F \,, \tag{A-5}$$

$$E\left(\phi_4^2\;\phi_2^2\right) = E\left(x_4^2\; x_0^2 [x_3\, x_1 + x_2 (x_0 + x_4)]^2\right) =$$

$$= E\left(x_4^2\; x_0^2\left[x_3^2\, x_1^2 + x_2^2 (x_0 + x_4)^2 + 2\, x_3\, x_2\, x_1 (x_0 + x_4)\right]\right) =$$

$$= 1 + (F + F) = 2F + 1 \; , \tag{A-6}$$

$$E\left(\phi_4^2 \; \phi_1^2\right) = E\left(x_4^2 \; x_0^2[x_1(x_0 + x_2) + x_3(x_2 + x_4)]^2\right) =$$

$$= E\left(x_4^2 \; x_0^2\left[x_1^2(x_0+x_2)^2 + x_3^2(x_2+x_4)^2 + 2 \; x_3 \; x_1(x_0+x_2)(x_2+x_4)\right]\right) =$$

$$= (F + 1) + (1 + F) = 2F + 2 \; , \tag{A-7}$$

$$E\left(\phi_3^4\right) = E\left([x_3 \; x_0 + x_4 \; x_1]^4\right) = F^2 + 6 + F^2 = 2F^2 + 6 \; , \tag{A-8}$$

$$E\left(\phi_3^2 \; \phi_2^2\right) = E\left([x_3 \; x_0 + x_4 \; x_1]^2 \; [x_3 \; x_1 + x_2(x_4 + x_0)]^2\right) =$$

$$= E\left(\left[x_3^2 \; x_0^2 + x_4^2 \; x_1^2 + 2 \; x_4 \; x_3 \; x_1 \; x_0\right]\left[x_3^2 \; x_1^2 + x_2^2(x_4 + x_0)^2 + \right.\right.$$

$$\left.\left. + 2 \; x_3 \; x_2 \; x_1(x_4 + x_0)\right]\right) = F + (1+F) + F + (F+1) = 4F + 2 \; , \tag{A-9}$$

$$E\left(\phi_3^2 \; \phi_1^2\right) = E\left([x_3 \; x_0 + x_4 \; x_1]^2 \; [x_1(x_2 + x_0) + x_3(x_4 + x_2)]^2\right) =$$

$$= E\left(\left[x_3^2 \; x_0^2 + x_4^2 \; x_1^2 + 2 \; x_4 \; x_3 \; x_1 \; x_0\right]\left[x_1^2(x_2 + x_0)^2 + x_3^2(x_4 + x_2)^2 + \right.\right.$$

$$\left.\left. + 2 \; x_3 \; x_1(x_2 + x_0)(x_4 + x_2)\right]\right) =$$

$$= (1 + F) + F(1 + 1) + F(1 + 1) + (F + 1) + 4 = 6F + 6 \; , \tag{A-10}$$

$$E\left(\phi_2^4\right) = E\left([x_3 \; x_1 + x_2(x_4 + x_0)]^4\right) =$$

$$= F^2 + 6(1 + 1) + F(F + 6 + F) = 3F^2 + 6F + 12, \tag{A-11}$$

$$E\left(\phi_2^2\ \phi_1^2\right) = E\left([x_3\ x_1 + x_2(x_4 + x_0)]^2\ [x_1(x_2 + x_0) + x_3(x_4 + x_2)]^2\right)$$

$$= E\left(\left[x_3^2\ x_1^2 + x_2^2(x_4 + x_0)^2 + 2\ x_3\ x_2\ x_1(x_4 + x_0)\right] \times\right.$$

$$\left.\times\ \left[x_1^2(x_2 + x_0)^2 + x_3^2(x_4 + x_2)^2 + 2\ x_3\ x_1(x_2 + x_0)(x_4 + x_2)\right]\right) =$$

$$= F(1 + 1) + F(1 + 1) +$$

$$+ E\left(\left[x_4^2 + 2\ x_4\ x_0 + x_0^2\right]\left[x_2^4 + 2\ x_2^3\ x_0 + x_2^2\ x_0^2\right]\right) +$$

$$+ E\left(\left[x_4^2 + 2\ x_4\ x_0 + x_0^2\right]\left[x_4^2\ x_2^2 + 2\ x_4\ x_2^3 + x_2^4\right]\right) +$$

$$+ 4\ E\left(x_2(x_4 + x_0)(x_2 + x_0)(x_4 + x_2)\right) =$$

$$= 4F + (F + 1 + F + F) + (F + F + 1 + F) + 4(1 + 1) = 10F + 10, \qquad \text{(A-12)}$$

$$E\left(\phi_1^4\right) = E\left([x_1(x_2 + x_0) + x_3(x_4 + x_2)]^4\right) =$$

$$= E\left(x_1^4(x_2 + x_0)^4 + 6\ x_1^2(x_2 + x_0)^2\ x_3^2(x_4 + x_2)^2 + x_3^4(x_4 + x_2)^4\right) =$$

$$= F(F + 6 + F) + 6\ E\left(\left[x_2^2 + 2\ x_2\ x_0 + x_0^2\right]\left[x_4^2 + 2\ x_4\ x_2 + x_2^2\right]\right) +$$

$$+ F(F + 6 + F) = 4F^2 + 12F + 6(1 + F + 1 + 1) =$$

$$= 4F^2 + 18F + 18\ . \qquad \text{(A-13)}$$

Now, we combine all the component averages, above, to obtain mean square value

$$E\left(W_5^2\right) = \frac{8}{625}(5F^2 + 38F + 39) \qquad \text{(A-14)}$$

and variance

$$V_5 = \frac{8}{625}(5F^2 + 38F - 11) \ . \qquad \text{(A-15)}$$

SPECIAL CASE K = 6

Now, we adopt a very useful shorthand notation to handle the rest of the cases of interest. For example, here, $\phi_5 = x_5\ x_0$ and $\phi_5^2 = x_5^2\ x_0^2$, which is denoted by 5500; that is, the superfluous x is ignored when possible. Also, $x_4\ x_2^2\ x_0$ is denoted by 4220. With this background, we now have

$$\phi_5^2 = 5500 \ , \quad \phi_4^2 = 4400+5511+2(5410) \ ,$$

$$\phi_3^2 = 3300+4411+5522+2(4310+5320+5421) \ ,$$

$$\phi_2^2 = 2200+3311+4422+5533+2(3210+4220+5320+4321+5331+5432) \ ,$$

$$\phi_1^2 = 1100+2211+3322+4433+5544+$$

$$+2(2110+3210+4310+5410+3221+4321+5421+4332+5432+5443) \ . \qquad \text{(A-16)}$$

From (13), there follows

$$W_6 = \frac{2}{36}\sum_{n=1}^{5} \phi_n^2 = \frac{1}{18}\left(\phi_1^2 + \phi_2^2 + \cdots + \phi_5^2\right) \qquad \text{(A-17)}$$

and

$$324\ W_6^2 = \phi_1^4+\phi_2^4+\phi_3^4+\phi_4^4+\phi_5^4 + 2\left(\phi_1^2\phi_2^2 + \cdots + \phi_4^2\phi_5^2\right) . \tag{A-18}$$

We also abbreviate the following ensemble averages as follows

$$E\left(\phi_m^2\ \phi_n^2\right) = Tmn . \tag{A-19}$$

Then, there follows, in a straightforward but tedious manner,

$$T55 = F^2 , \quad T54 = F+F = 2F , \quad T53 = F+1+F = 2F+1 ,$$
$$T52 = F+1+1+F = 2F+2 , \quad T51 = F+1+1+1+F = 2F+3 ,$$
$$T44 = F^2+6+F^2 = 2F^2+6 , \quad T43 = (F+F+1)+(1+F+F) = 4F+2 ,$$
$$T42 = (F+1+F+1)+(1+F+1+F) = 4F+4 ,$$
$$T41 = (F+1+1+F+F)+(F+F+1+1+F)+4 = 6F+8 ,$$
$$T33 = 3F^2+4(1+1+1)+2(1+1+1) = 3F^2+18 ,$$
$$T32 = (F+F+1+F)+(1+F+F+1)+(F+1+F+F)+4(1) = 8F+8 ,$$
$$T31 = (F+1+F+F+1)+(F+F+1+F+F)+(1+F+F+1+F)+4(1+1) = 10F+13 ,$$
$$T22 = 4F^2+4(1+F+1+1+F+1)+2(1+F+1+1+F+1) = 4F^2+12F+24 ,$$
$$T21 = (F+F+F+1+1)+(F+F+F+F+1)+(1+F+F+F+F)+(1+1+F+F+F)+12 = 14F+18$$
$$T11 = 5F^2+(4+2)(F+1+1+1+F+1+1+F+1+F) = 5F^2+24F+36 . \tag{A-20}$$

The desired average is, from (A-18) - (A-20),

$$324\ E\left(W_6^2\right) = 15F^2 + 144F + 202 . \tag{A-21}$$

The variance of W_6 is then

$$V_6 = \frac{1}{324}\left(15F^2 + 144F - 23\right) . \tag{A-22}$$

SPECIAL CASE K = 7

Continuing in the fashion established above, we now have

$$\phi_6^2 = 6600 \ , \quad \phi_5^2 = 5500+6611+2(6510) \ ,$$

$$\phi_4^2 = 4400+5511+6622+2(5410+6420+6521) \ ,$$

$$\phi_3^2 = 3300+4411+5522+6633+2(4310+5320+6330+5421+6431+6532) \ ,$$

$$\phi_2^2 = 2200+3311+4422+5533+6644+$$

$$+2(3210+4220+5320+6420+4321+5331+6431+5432+6442+6543) \ ,$$

$$\phi_1^2 = 1100+2211+3322+4433+5544+6655+2(2110+3210+4310+5410+$$

$$+6510+3221+4321+5421+6521+4332+5432+6532+5443+6543+6554) \ . \quad \text{(A-23)}$$

From (13),

$$W_7 = \frac{2}{49}\left(\phi_1^2 + \phi_2^2 + \cdots + \phi_6^2\right) \quad \text{(A-24)}$$

and therefore

$$\frac{2401}{4} W_7^2 = \phi_1^4 + \cdots + \phi_6^4 + 2\left(\phi_1^2 \ \phi_2^2 + \cdots \ \phi_5^2 \ \phi_6^2\right) \ . \quad \text{(A-25)}$$

The required averages are as follows:

$$T66 = F^2 \ , \quad T65 = F+F = 2F \ , \quad T64 = F+1+F = 2F+1 \ ,$$

$$T63 = F+1+1+F = 2F+2 \ , \quad T62 = F+1+1+1+F = 2F+3 \ ,$$

$$T61 = F+1+1+1+1+F = 2F+4 \ , \quad T55 = F^2+F^2+4+2 = 2F^2+6 \ ,$$

$$T54 = (F+F+1)+(1+F+F) = 4F+2 \ , \quad T53 = (F+1+F+1)+(1+F+1+F) = 4F+4 \ ,$$

$$T52 = (F+1+1+F+1)+(1+F+1+1+F) = 4F+6 \ ,$$

$$T51 = (F+1+1+1+F+F)+(F+F+1+1+1+F)+4 = 6F+10 \ ,$$

$$T44 = 3F^2+4(1+1+1)+2(1+1+1) = 3F^2+18 \ ,$$

$$T43 = (F+F+1+1)+(1+F+F+1)+(1+1+F+F) = 6F+6 \ ,$$

$$T42 = (F+1+F+1+F)+(1+F+1+F+1)+(F+1+F+1+F)+4(1) = 8F+11 \ ,$$

$$T41 = (3F+3)+(4F+2)+(3F+3)+4(1+1) = 10F+16 \ ,$$

$$T33 = 4F^2+4(1+1+F+1+1+1)+2(1+1+F+1+1+1) = 4F^2+6F+30 \ ,$$

$$T32 = (3F+2)+(3F+2)+(3F+2)+(3F+2)+4(1+1) = 12F+16 \ ,$$

$$T31 = (3F+3)+(4F+2)+(4F+2)+(3F+3)+4(1+1+1) = 14F+22 \ ,$$

$$T22 = 5F^2+(4+2)(1+F+1+1+1+F+1+1+F+1) = 5F^2+18F+42 \ ,$$

$$T21 = 2(3F+3)+3(4F+2)+4(1+1+1+1) = 18F+28 \ ,$$

$$T11 = 6F^2+(4+2)(F+1+1+1+1+F+1+1+1+F+1+1+F+1+F) = 6F^2+30F+60 \ . \tag{A-26}$$

The average of interest is, from (A-25) and (A-26),

$$\frac{2401}{4} E\left(W_7^2\right) = 21F^2 + 246F + 418 \ , \tag{A-27}$$

leading to variance

$$V_7 = \frac{4}{2401}\left(21F^2 + 246F - 23\right) \ . \tag{A-28}$$

SPECIAL CASE K = 8

This is the last case that we need to evaluate. We now have

$$\phi_7^2 = 7700\ , \quad \phi_6^2 = 6600+7711+2(7610)\ ,$$

$$\phi_5^2 = 5500+6611+7722+2(6510+7520+7621)\ ,$$

$$\phi_4^2 = 4400+5511+6622+7733+2(5410+6420+7430+6521+7531+7632)\ ,$$

$$\phi_3^2 = 3300+4411+5522+6633+7744+$$

$$+2(4310+5320+6330+7430+5421+6431+7441+6532+7542+7643)\ ,$$

$$\phi_2^2 = 2200+3311+4422+5533+6644+7755+2(3210+4220+5320+6420+$$

$$+7520+4321+5331+6431+7531+5432+6442+7542+6543+7553+7654)\ ,$$

$$\phi_1^2 = 1100+2211+3322+4433+5544+6655+7766+$$

$$+2(2110+3210+4310+5410+6510+7610+3221+4321+5421+6521+7621+$$

$$+4332+5432+6532+7632+5443+6543+7643+6554+7654+7665)\ . \qquad \text{(A-29)}$$

From (13) again,

$$W_8 = \frac{2}{64}\left(\phi_1^2 + \phi_2^2 + \cdots + \phi_7^2\right)\ , \qquad \text{(A-30)}$$

giving

$$1024\ W_8^2 = \phi_1^4 + \cdots + \phi_7^4 + 2\left(\phi_1^2\ \phi_2^2 + \cdots + \phi_6^2\ \phi_7^2\right)\ . \qquad \text{(A-31)}$$

The averages needed are listed below.

$$T77 = F^2\ , \quad T76 = 2F\ , \quad T75 = 2F+1\ , \quad T74 = 2F+2\ ,$$

$$T73 = 2F+3\ , \quad T72 = 2F+4\ , \quad T71 = 2F+5\ ,$$

$$T66 = F^2+F^2+4+2 = 2F^2+6 , \quad T65 = (F+F+1)+(1+F+F) = 4F+2 ,$$

$$T64 = (F+1+F+1)+(1+F+1+F) = 4F+4 ,$$

$$T63 = (F+1+1+F+1)+(1+F+1+1+F) = 4F+6 ,$$

$$T62 = (F+1+1+1+F+1)+(1+F+1+1+1+F) = 4F+8 ,$$

$$T61 = (3F+4)+(3F+4)+4 = 6F+12 ,$$

$$T55 = 3F^2+4(1+1+1)+2(1+1+1) = 3F^2+18 ,$$

$$T54 = (F+F+1+1)+(1+F+F+1)+(1+1+F+F) = 6F+6 ,$$

$$T53 = 3(2F+3) = 6F+9 ,$$

$$T52 = (3F+3)+(2F+4)+(3F+3)+4(1) = 8F+14 ,$$

$$T51 = (3F+4)+(4F+3)+(3F+4)+4(1+1) = 10F+19 ,$$

$$T44 = 4F^2+4(6)+2(6) = 4F^2+36 ,$$

$$T43 = (3F+2)+(2F+3)+(2F+3)+(3F+2)+4(1) = 10F+14 ,$$

$$T42 = 4(3F+3)+4(1+1) = 12F+20 ,$$

$$T41 = 2(3F+4)+2(4F+3)+4(1+1+1) = 14F+26 ,$$

$$T33 = 5F^2+4(2F+8)+2(2F+8) = 5F^2+12F+48 ,$$

$$T32 = 4(3F+3)+(4F+2)+4(1+1+1) = 16F+26 ,$$

$$T31 = 2(3F+4)+3(4F+3)+4(1+1+1+1) = 18F+33 ,$$

$$T22 = 6F^2+4(4F+11)+2(4F+11) = 6F^2+24F+66 ,$$

$$T21 = 2(3F+4)+4(4F+3)+4(1+1+1+1+1) = 22F+40 ,$$

$$T11 = 7F^2+4(6F+15)+2(6F+15) = 7F^2+36F+90 . \tag{A-32}$$

The desired average is therefore

$$1024 \, E\left(W_8^2\right) = 28F^2 + 384F + 772 , \tag{A-33}$$

giving variance

$$V_8 = \frac{1}{256}\left(7F^2 + 96F - 3\right) . \tag{A-34}$$

APPENDIX B. PROGRAM FOR ESTIMATION OF DISTRIBUTIONS OF W_K

```
10    T=1E6                      !  NUMBER OF TRIALS   "NUWC TR10237"
20    K=32                       !  NUMBER OF DATA POINTS, ARBITRARY
30    M=64                       !  FFT SIZE, M >= 2K-1, POWER OF 2
40    L=11000                    !  NUMBER OF LEVELS FOR DISTRIBUTION
50    Dw=.001                    !  INCREMENT IN W
60    Gr=1000                    !  GRID SPACING
70      PRINTER IS PRT
80    PRINT "K =";K;"   T =";T;"   Dw =";Dw;"   UNIFORM"
90      PRINTER IS CRT
100   DOUBLE T,K,M,L,M1,Ms,M2,K1,Ts,Ks   !  INTEGERS, NOT DP
110   DIM Cos(512),X(2048),Y(2048),V(30000)
120   M1=M-1
130   REDIM Cos(0:M/4),X(0:M1),Y(0:M1),V(0:L)
140   A=2.*PI/M
150   FOR Ms=0 TO M/4
160   Cos(Ms)=COS(A*Ms)          !  QUARTER-COSINE TABLE IN Cos(*)
170   NEXT Ms
180   M2=M/2
190   K1=K-1
200   T1=1./T
210   F=12./(K*M)                !  UNIT-VARIANCE UNIFORM
220   F=F*F                      !  RANDOM VARIABLES (x(subk))
230   Mu=K1/K                    !  EXACT MEAN OF WK
240   Mu1=Var=0.
250     Ta=TIMEDATE
260   FOR Ts=1 TO T
270   FOR Ks=0 TO K1
280   X(Ks)=RND-.5               !  ZERO MEAN
290   Y(Ks)=0.                   !  REAL INPUT
300   NEXT Ks
310   FOR Ks=K TO M1
320   X(Ks)=Y(Ks)=0.
330   NEXT Ks
340   CALL Fft14(M,Cos(*),X(*),Y(*))
350   S2=S4=0.
360   FOR Ms=1 TO M2-1           !  ZERO TO NYQUIST
370   X=X(Ms)
380   Y=Y(Ms)
390   A=X*X+Y*Y
400   S2=S2+A
410   S4=S4+A*A
420   NEXT Ms
430   X=X(0)
440   A=X(M2)
450   X=X*X
460   A=A*A
470   S2=X+A+2.*S2
480   S4=X*X+A*A+2.*S4
490   W=F*(M*S4-S2*S2)           !  WHITENESS MEASURE WK
```

```
500     Mu1=Mu1+W
510     Var=Var+(W-Mu)*(W-Mu)!  USE KNOWN MEAN Mu
520     Ms=INT(W/Dw)
530     Ms=MIN(Ms,L)
540     V(Ms)=V(Ms)+T1          !  INCREMENTAL PROBABILITIES
550     NEXT Ts
560       Tb=TIMEDATE
570       PRINTER IS PRT
580       PRINT (Tb-Ta)/3600;" HOURS"
590       PRINT
600     Mu1=Mu1/T               !  ESTIMATED MEAN OF WK
610     Var=Var/T               !  ESTIMATED VARIANCE OF WK
620     PRINT "Mu1 =";Mu1;"       Mu =";Mu
630     PRINT "Var =";Var
640     PRINT
650     PLOTTER IS "GRAPHICS"
660     GRAPHICS ON
670     WINDOW 0,L,-6,0
680     LINE TYPE 3
690     GRID Gr,1
700     LINE TYPE 1
710     C=0.
720     FOR Ms=0 TO L-1
730     C=C+V(Ms)               !  CDF OF WHITENESS MEASURE WK
740     IF C>0. THEN 760
750     GOTO 770
760     PLOT Ms+1,LGT(C)
770     NEXT Ms
780     PENUP
790     E=S1=S2=0.
800     FOR Ms=L TO 1 STEP -1
810     E=E+V(Ms)               !  EDF OF WHITENESS MEASURE WK
820     S1=S1+E
830     S2=S2+S1
840     IF E>0. THEN 860
850     GOTO 870
860     PLOT Ms,LGT(E)
870     NEXT Ms
880     PLOT 0,0
890     PENUP
900     Mu1=Dw*(.5+S1)          !  ESTIMATED MEAN OF WK
910     Mu2=2.*Dw*Dw*S2         !  SEE APPENDIX C
920     PRINT "Mu1 =";Mu1;"       Mu =";Mu
930     PRINT "Var =";Mu2-Mu*Mu ! ESTIMATED VARIANCE OF WK
940     PRINT
950       PRINTER IS CRT
960     PAUSE
970     END
980     !
990     SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! N<=2^14=16384; 0 SUBS
```

APPENDIX C. EVALUATION OF MOMENTS DIRECTLY FROM MEASURED EXCEEDANCE DISTRIBUTION

Let x be a positive random variable with probability density function p, cumulative distribution function (CDF) C, and exceedance distribution function (EDF) E. Let the measurements of these distributions be the interval probabilities

$$V_n = \text{Prob}(n\Delta \le x < (n+1)\Delta) \quad \text{for } 0 \le n \, . \tag{C-1}$$

Then

$$1 = \int_0^\infty dx \; p(x) = \sum_{n=0}^{\infty} \int_{n\Delta}^{(n+1)\Delta} dx \; p(x) = \sum_{n=0}^{\infty} V_n \, . \tag{C-2}$$

At the same time, we can express

$$V_n = C((n+1)\Delta) - C(n\Delta) = E(n\Delta) - E((n+1)\Delta) \, , \tag{C-3}$$

which can be inverted, leading respectively to EDF and CDF

$$E(n\Delta) = \text{Prob}(x \ge n\Delta) = \sum_{m=n}^{\infty} V_m \quad \text{for } n \ge 0 \, , \tag{C-4}$$

$$C(n\Delta) = \text{Prob}(x < n\Delta) = \sum_{m=0}^{n-1} V_m \quad \text{for } n \ge 1 \, . \tag{C-5}$$

There also follows

$$E(0) = 1 \, , \quad E((n+1)\Delta) = E(n\Delta) - V_n \quad \text{for } n \ge 0 \, , \tag{C-6}$$

or, as an alternative form to (C-4) if desired,

$$E(\Delta) = 1-V_0 \, , \quad E(2\Delta) = 1-V_0-V_1 \, , \quad E(3\Delta) = 1-V_0-V_1-V_2 \, , \ldots \tag{C-7}$$

The first two moments of random variable x can be developed as

$$\mu_1 = \int_0^\infty dx\ x\ p(x) = \int_0^\infty dx\ E(x) \cong \Delta\left[\frac{1}{2}\ E(0) + \sum_{n=1}^{\infty} E(n\Delta)\right] , \qquad \text{(C-8)}$$

and

$$\mu_2 = \int_0^\infty dx\ x^2\ p(x) = 2 \int_0^\infty dx\ x\ E(x) \cong 2\ \Delta^2 \sum_{n=1}^{\infty} n\ E(n\Delta) . \qquad \text{(C-9)}$$

These results can be rapidly evaluated by recursion. For $E((N+1)\Delta) = 0$, use

```
E=S1=S2=0.
FOR Ns=N TO 1 STEP -1
E=E+V(Ns)
S1=S1+E
S2=S2+S1
NEXT Ns
Mu1=Delta*(.5+S1)
Mu2=2.*Delta*Delta*S2
```

(C-10)

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